Comments on 2.12e

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}$$ continued fraction

$$X_1 = \frac{1}{2}$$

$$X_{n+1} = \frac{1}{2+X_n} \Rightarrow X_{n+2} = \frac{1}{2+X_{n+1}} = \frac{1}{2+\frac{1}{2+X_n}}$$

→ Not monotone, but we have to prove it converges.

But.... the subsequences (X_{2n}) and (X_{2n-1}) are monotone.

Suggestion: consider as 2 separate questions — evens & odd sequences.

⇒ Bdd & monotone — use MCT to show they converge, L thing & OLT to find the limit. → should be same.

<u>Thus</u>: $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$,

$$|x_{2n} - L| < \varepsilon.$$

also, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$,

$$|x_{2n-1} - L| < \varepsilon.$$

Let $N_3 = \max\{2N_1, 2N_2\}$

Then $\forall k \geq N_3$,

$$|x_k - L| < \varepsilon.$$

Another Tool

Cauchy Criterion (CC)

A sequence (x_n) converges

$\iff \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t.

$\forall n, k \geq N, \ |x_n - x_k| < \varepsilon.$

Extremely useful: we don't need to know what L is!

Scratch: ($\Longrightarrow$) If $L = \lim x_n$

$$|x_n - x_k| = |(x_n - L) - (x_k - L)|$$

$$\leq |x_n - L| + |x_k - L| < \tfrac{\varepsilon}{2} + \varepsilon/2$$

Proof of $\Rightarrow$: Suppose $\lim x_n = L$. Then,

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|x_n - L| < \frac{\varepsilon}{2}.$$

Then, if $n, k \geq N$, we have

$$|x_n - x_k| = |(x_n - L) - (x_k - L)| \leq |x_n - L| + |x_k - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ CC is satisfied.

Sketch of proof of $\Leftarrow$: Suppose CC is satisfied.

① Show x_n must be bounded.

$$\varepsilon > |x_n - x_k| \geq |x_n| - |x_k|$$

fix this.

② let $L = \limsup \{x_n\}$.

show that x_n actually converges to this L.

Example: We can use this in studying series convergence:

Sp. $\sum a_k$ is a series, so

it corresponds to the sequence (S_n),

where $S_n = \sum_{k=1}^{n} a_k$.

Using CC on (S_n)

$\Rightarrow$ $\sum a_k$ converges

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$\forall n > k \geq N$,

$|S_n - S_k| < \varepsilon$.

$$\Leftrightarrow \quad \left| \sum_{j=1}^{n} a_j - \sum_{j=1}^{k} a_j \right| < \varepsilon$$

$$\Leftrightarrow \quad \left| \sum_{j=k+1}^{n} a_j \right| < \varepsilon.$$

CC for series is often stated

as: $\sum a_k$ converges

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > k \geq N$,

$$\left| \sum_{j=k}^{n} a_j \right| < \varepsilon.$$

often called the "tail" of the series.

Again, this is useful, since then we don't need to know what the series converges to.

Example

Exercise 2.5.1. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$, and no subsequences converging to points outside of this set.

(a) (a_n) sequence, (a_{n_k}) subseq. bdd.
No subseq that converges. Impossible.
By BW, (a_{n_k}) must have a subseq that converges, which is also a subsequence of (a_n).

(b) $\left(\frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{5}{4}, \frac{1}{5}, \frac{6}{5}, \ldots\right)$

(c) eg. write $\mathbb{Q} = \{q_1, q_2, \ldots\}$ ← can actually find a subseq. converging to every real #!

$(1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$

(d) Impossible: must have subseq. conv. to 0.

$\exists a_{n_1}$ s.t. $|a_{n_1} - \frac{1}{2}| < \frac{1}{2}$.

$$\Rightarrow |a_{n_1}| - \frac{1}{2} < \frac{1}{2}$$

$$|a_{n_1}| < 1$$

Choose $n_2 > n_1$ s.t.

$$|a_{n_2} - \frac{1}{4}| < \frac{1}{4}$$

$$\Rightarrow |a_{n_2}| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Choose $n_3 > n_2$ s.t.

$$|a_{n_3} - \frac{1}{8}| < \frac{1}{8}$$

$$\Rightarrow |a_{n_3}| < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

$\vdots$

$(a_{n_j})_{j \geq 1}$ conv. to 0.

Exercise 2.5.2. Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

(a) True every (x_{n_j}) converges if $(n_j) \neq (1,2,3,\dots)$

example: $(x_2, x_3, x_4, \dots) \leftarrow$ "if this converges,

then $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $j \geq N$

then $|x_{j+1} - L| < \varepsilon$

$\uparrow$ satisfies limit defn

~~N~~ $N+1$.

(b) True Related to

"Every subseq of a convergent seq. converges to the same limit".

So if $\exists$ divergent subseqn, then the orig sequence can't converge.

(c) True Choose one subseq converging to the lim sup, one conv. to lim inf. If they were equal, the original seq. would have converged.

(d) True If (x_n) is monotone & diverges, it's unbounded $\leftarrow$ use this to show subseq is unbdd.

(Because if subseq. is bdd, so is the original.

x_{n_k} $\quad n_k > k$

increasing $\quad x_{n_k} \geq x_k$

if $M \geq x_{n_k} \geq x_k \quad \forall k$)

Exercise 2.6.2. Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

(a) Possible: $\left(\frac{(-1)^n}{n}\right) \to 0$.

(b) Impossible: Every subseq of a conv. seq converges to the same limit.

(c) Divergent + montone $\Rightarrow$ unbdd.
Imp.
$\Rightarrow$ every subseq is unbdd.
$\Rightarrow$ every subseq diverges.

(d) Possible $(1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \ldots)$
contains the Cauchy subsequence $(1, 1, 1, 1, 1, \cdots)$